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Spin-$\frac{1}{2}$ lattice systems: a new approach to duality transformations

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Abstract. A straightforward algebraic technique is introduced in order to establish duality relations between spin-$\frac{1}{2}$ lattice systems. A new self-duality relation is found.

The duality transformations were put forward by Kramers and Wannier (1941) for the square Ising model using the transfer matrix formalism.

Onsager, by introducing the dual lattice, gave a topological picture of the dual transformations (Wannier 1945) and thus they were extended to varieties of two-dimensional lattices.

No major progress occurred until the early 1970s. During 1971 Wegner established duality relations for generalised Ising models. However he does not provide a simple method for explicitly constructing these dual theories. Afterwards Merlini and Gruber (1972) presented another derivation of such transformations, using a mathematical formalism based on the group properties of general lattice systems.

In the present paper we introduce a new method for deriving duality relations. It is a very simple algebraic procedure applicable to generalised Ising models (we call a generalised Ising model a set of Ising variables interacting with an arbitrary Hamiltonian in any spatial dimension).

We will not work in the general case but show the technique applying it in some usual models. The general procedure will be easily introduced by the examples considered in the following.

In all the cases we will consider a lattice with periodic boundary conditions.

(a) Ising model with external field in a square lattice. The partition function is

$$Z=\sum_{s}\exp\left(\sum_{x}K_1s_x+K_2s_x(s_{x+\mu}+s_{x+\nu})\right) \qquad (1)$$

where $K_1=h/kT$ and $K_2=J/kT$, x indicates a generic point of the lattice and μ and ν are unit vectors in the horizontal and vertical directions respectively.

As is well known (1) can be written as

$$Z=[\cosh(K_1)\cosh^2(K_2)]^N\sum_{s}\prod_{x}(1+\alpha_1s_x)(1+\alpha_2s_xs_{x+\mu})(1+\alpha_2s_xs_{x+\nu}) \qquad (2)$$

with $\alpha_1=\tanh(K_1)$, $\alpha_2=\tanh(K_2)$, and where N is the number of lattice sites.

The first step of our method consists in expanding the products that contain the interactions, by introducing variables which take the values zero and one (Savit 1980).

In the present case two variables of this type are needed. It leads to

$$Z=[\cosh(K_1)\cosh^2(K_2)]^N \sum_{s,n,m}\prod_x (1+\alpha_1 s_x)\alpha_2^{n_x+m_x} s_x^{n_x+m_x+n_{x-\mu}+m_{x-\nu}}. \tag{3}$$

The s variable is decoupled in (3) and it can be summed up, giving

$$Z=[2\cosh(K_1)\cosh^2(K_2)]^N \sum_{n,m}\prod_x \alpha_2^{n_x+m_x}[\delta_2(n_x+m_x+n_{x-\mu}+m_{x-\nu})$$
$$+\alpha_1\delta_2(n_x+m_x+n_{x-\mu}+m_{x-\nu}+1)] \tag{4}$$

where $\delta_2(n)$ is a Kronecker delta function modulo two. It is zero if n is odd and one if n is even.

Now, instead of solving the constraint imposed by the delta functions (Savit 1980), we express them as follows:

$$\begin{aligned}\delta_2(n_x+m_x+n_{x-\mu}+m_{x-\nu}) &= \tfrac{1}{2}(s_x^{(1)}s_x^{(2)}s_{x-\mu}^{(1)}s_{x-\nu}^{(2)}+1)\\ \delta_2(n_x+m_x+n_{x-\mu}+m_{x-\nu}+1) &= \tfrac{1}{2}(1-s_x^{(1)}s_x^{(2)}s_{x-\mu}^{(1)}s_{x-\nu}^{(2)})\end{aligned} \tag{5}$$

where we have introduced Ising variables according to

$$s_x^{(1)}=2n_x-1 \qquad s_x^{(2)}=2m_x-1. \tag{6}$$

Therefore, by using the identity

$$\prod_x \alpha^{n_x+m_x}=\prod_x [1+n_x(\alpha-1)][1+m_x(\alpha-1)] \tag{7}$$

and taking into account (5) we obtain

$$Z=[\tfrac{1}{4}\exp(K_1+2K_2)]^N \sum_{s^{(1)},s^{(2)}}\prod_x (1+v_2 s_x^{(1)})(1+v_2 s_x^{(2)})(1+v_1 s_x^{(1)}s_{x+\mu}^{(1)}s_x^{(2)}s_{x+\nu}^{(2)}) \tag{8}$$

where we have changed the signs of the unit vectors μ and ν, based on the symmetry of the lattice. Moreover we have called

$$\begin{aligned} v_1 &= \frac{1-\alpha_1}{1+\alpha_1}=\exp(-2K_1)\\ v_2 &= \frac{1-\alpha_2}{1+\alpha_2}=\exp(-2K_2).\end{aligned} \tag{9}$$

If we associate the variable $s^{(1)}$ with the vertical links of the lattice and $s^{(2)}$ with the horizontal ones, the well known duality relation between the Ising model with external field and its gauge invariant version is obtained (Wegner 1971, Balian *et al* 1975). In particular, in the limit $h\to 0$ the Kramers-Wannier self-duality relation results (Balian *et al* 1975).

The fundamental fact of the above calculations has been to associate a new Ising variable at each interacting monomial term of the original Hamiltonian.

(*b*) The second model that we will analyse is the triangular lattice with two-body anisotropic interactions. Before we consider this model we will set up some basic notation. For each variable $K_i=J_i/kT$, we define a dual variable K_i^* by

$$\exp(-2K_i)=\tanh(K_i^*). \tag{10}$$

Besides, given three interaction coefficients K_1, K_2, K_3, we call

$$\varepsilon=[\tfrac{1}{8}\sinh(2K_1)\sinh(2K_2)\sinh(2K_3)]^{1/2}. \tag{11}$$

From K_1^*, K_2^*, K_3^*, we define R, M_1, M_2, M_3 such that if a, b, c, d are any four spins with values ± 1

$$\sum_d \exp[-(K_1^* a + K_2^* b + K_3^* c)d] = R \exp(M_1 bc + M_2 ca + M_3 ab) \tag{12}$$

where

$$\begin{aligned}
e^{4M_1} &= \frac{\cosh(K_1^* + K_2^* + K_3^*)\cosh(-K_1^* + K_2^* + K_3^*)}{\cosh(K_1^* - K_2^* + K_3^*)\cosh(K_1^* + K_2^* - K_3^*)} \\
e^{4M_2} &= \frac{\cosh(K_1^* + K_2^* + K_3^*)\cosh(-K_2^* + K_1^* + K_3^*)}{\cosh(K_2^* - K_1^* + K_3^*)\cosh(K_1^* + K_2^* - K_3^*)} \\
e^{4M_3} &= \frac{\cosh(K_1^* + K_2^* + K_3^*)\cosh(-k_3^* + k_1^* + k_2^*)}{\cosh(K_3^* - K_2^* + K_1^*)\cosh(K_3^* + K_2^* - K_1^*)} \\
R &= 2\exp(-M_1 - M_2 - M_3)\cosh(K_1^* + K_2^* + K_3^*).
\end{aligned} \tag{13}$$

These expressions are the algebraic form of the 'star-triangle' relation (Wannier 1945, Baxter 1982), and will be useful in the following.

For brevity we will set

$$\begin{aligned}
s_x^{(i)} &= s_i & s_{x+\mu}^{(i)} &= s_i' \\
s_{x+\nu}^{(i)} &= s_i'' & s_{x+\mu+\nu}^{(i)} &= s_i'''
\end{aligned} \tag{14}$$

as shown in figure 1.

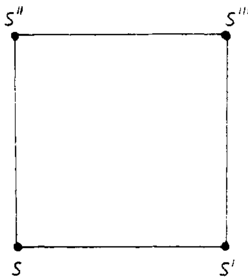

Figure 1. s, s', s'', s''' are four spins round a face of the square lattice.

The partition function of the triangular lattice Ising model is

$$Z = \sum_s \exp\left(\sum_x K_1 ss' + K_2 ss'' + K_3 s's''\right) \tag{15}$$

where we have considered the triangular lattice as a square one with diagonal bonds.

The above procedure leads to

$$Z = \varepsilon^N \sum_{s_1, s_2, s_3} \prod_x \exp(-K_1^* s_1 - K_2^* s_2 - K_3^* s_3)(1 + s_1 s_1' s_2 s_2'' s_3' s_3''). \tag{16}$$

In consequence we must solve the constraint $s_1 s_1' s_2 s_2'' s_3' s_3'' = 1$. With the aim of simplifying it we introduce the following change of variables: $s_1 \to s_1 s_3$, $s_2 \to s_2 s_3$. Then the summand in (16) becomes

$$\prod_x \exp(-K_1^* s_1 s_3 - K_2^* s_2 s_3 - K_3^* s_3)(1 + s_1 s_1' s_2 s_2''). \tag{17}$$

Since s_3 is decoupled, we can sum independently over each s_3 spin. Therefore, using the star-triangle relation above with

$$\begin{aligned} a &= s_1 \qquad & c &= 1 \\ b &= s_2 \qquad & d &= s_3 \end{aligned} \tag{18}$$

we obtain

$$Z = (\varepsilon R)^N \sum_{s_1, s_2} \prod_x \exp(M_1 s_2 + M_2 s_1 + M_3 s_1 s_2)(1 + s_1 s_1' s_2 s_2''). \tag{19}$$

Now, the spin set s_1, s_2 must satisfy the constraint

$$s_1 s_1' s_2 s_2'' = 1. \tag{20}$$

The general solution of (20) is (Balian *et al* 1975)

$$s_1 = ss'' \qquad s_2 = ss' \tag{21}$$

where s is also an Ising variable. In consequence

$$Z = (2R\varepsilon)^N \sum_s \exp\left(\prod_x M_1 ss' + M_2 ss'' + M_3 s's''\right). \tag{22}$$

Therefore, the well known self-duality relation for the triangle lattice Ising model is obtained, but without the intermediate step that transforms to the honeycomb lattice (Syozi 1972, Baxter 1982). This fact shows the simplicity of our method.

(c) The last example that we will consider is a model on a square lattice, with four spin interactions, an alternate three spin interactions and an external field.

The partition function is

$$Z = \sum_s \exp\left(\sum_x K_1 s + K_2 s' s'' s''' + K_3 s s' s'' s'''\right). \tag{23}$$

The cases $K_2 = 0$ or $K_3 = 0$ have been analysed by Merlini and Gruber (1972). They showed that both models are self-dual. Using the same procedure as above we obtain

$$Z = (2\varepsilon)^N \sum_{s_1, s_2} \prod_x \exp[-(K_2^* s_1 + K_3^* s_2 + K_1^* s_1' s_1'' s_1''' s_2 s_2' s_2'' s_2''')]. \tag{24}$$

As in (a), there is no constraint over the spin variables when an external field is present.

Changing the variables as follows

$$s_2 \to s_2 s_1 \tag{25}$$

the summand in (24) becomes

$$\prod_x \exp[-(K_2^* s_1 + K_3^* s_1 s_2 + K_1^* s_1 s_2 s_2' s_2'' s_2''')]. \tag{26}$$

Now the spin s_1 is decoupled. Summing it up for each site, and using again the star-triangle transformation, we obtain

$$Z = (2R\varepsilon)^N \sum_{s_2} \exp\left(\sum_x M_1 s_2 + M_2 s_2' s_2'' s_2''' + M_3 s_2 s_2' s_2'' s_2'''\right). \tag{27}$$

Therefore the model is self-dual. This result is new to our knowledge. It is interesting to point out that this duality relation is the same as the relation (22) for the triangular Ising model.

When $K_2 = 0(M_2 = 0)$ or $K_3 = 0(M_3 = 0)$ we recover the results of Merlini and Gruber (1972). For the case $K_1 = 0(M_1 = 0)$ the model is self-dual too. However, Merlini and Gruber (1972) affirm that the square Ising model and the two-dimensional triangular lattice with three-body nearest-neighbour interactions are the only self-dual system in two dimensions without an external field.

Our results disagree with this statement.

It is evident from the examples given that the method can be applied to any generalised Ising model, even if position-dependent interactions and external fields are present.

All the known self-duality relations can be achieved with this technique. Besides, in simple models, the form of the dual ones can be analysed by simple inspection of the original theory.

It is interesting to analyse three-dimensional systems with this procedure, because very few results about duality exist in these cases. Work in this direction is in progress.

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